Bilinear z-transformation and Pascal's triangle

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Abstract— Bilinear z-transformation is the most common method for converting the transfer function H(s) of the analog filter to the transfer function H (z) of the digital filter and vice versa. In this work, introducing the relationship between the digital coefficients and the analog coefficients in the matrix equation definitely involves the Pascal's triangle.

Index Terms—Bilinear z-transform, warping frequency, pre-warping frequency, frequency transformation, low pass to low pass, low pass to high pass, low pass to band pass, low pass to band stop, Pascal's triangle.

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1 INTRODUCTION

Since founded Pascal's triangle has proved to be a very useful applications in mathematics and in other fields. One of these wonderful applications is used to the bilinear z-transformation to convert from s-domain to z-domain and inverse from z-domain to s-domain. By using the Pascal's triangle, it can make easier to compute and hand-calculated the matrix equation of the analog and digital coefficients.

2 BILINEAR Z-TRANSFORMATION

The effective and popular method is currently used to convert an analog filter in s-domain into an equivalent digital filter in z-domain is the bilinear z-transform. This technique is one to one mapping the poles and zeros on the left half stable region in s-plane into inside unit circle in z-plane. The main advantage of this method is transform a stable designed analog filter to a stable digital filter which the frequency response has the same characteristics as frequency response of the analog filter. However, this method will give a non-linear relationship between analog frequency ω_A and digital frequency ω_D and leads to warping of digital frequency response.

2.1 Warping frequency and pre-warping frequency

The bilinear z-transform form the s-domain to z-domain is defined by

$$s = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}} \quad (1)$$

Where T is sampling period, substitute $z = e^{sT}$ and $s = j\omega$ into equation (1), will give the relationship between ω_A and ω_D as shown in equation (2) and this relationship is a non-linear [2] that causing by the 'tan' function and the sampling period T and this effects is called warping frequency.

$$\omega_A = \frac{2}{T} \tan\left(\frac{T}{2} \omega_D\right) \quad (2)$$

When converting an analog filter to a digital filter using bilinear z-transform method will give both filters have the same behaviour, but the behaviour is not matched at all the frequency in s-domain and digital domain as causing by the warping frequency. One way to overcome the warping frequency is called pre-warping frequency and it is expressed as in equation (3).

$$s = \frac{\omega_c}{\tan\left(\frac{T}{2}\omega_c\right)} \frac{1 - z^{-1}}{1 + z^{-1}} \quad (3)$$

Where ω_c is the pre-warping frequency or wanted frequency.

2.2 Bilinear z-transformation with pre-warping

The transfer function H(z) of the digital filers (low pass, high pass, band pass and band stop) can be obtained from the transfer function H(s) of a designed analog low pass filter. to design a digital filter, first transform a designed analog low pass filter to an analog filter which the same class of the digital filter using the frequency transformations [2],[4] and then apply the equation (3) bilinear z-transform with pre-warping.

The table.1 below, illustrated converting an analog low pass filter to a low pass, high pass, band pass and band stop digital filter using bilinear z-transform with pre-warping.

Table.1 Transform an analog low pass filter to a digital filter using bilinear z-transform with pre-warping.

	s-domain	Frequency transformations	z-domain
LP to LP	$H(s)$	$H\left(\dfrac{s}{\omega_c}\right)$	$H\left(c \dfrac{1 - z^{-1}}{1 + z^{-1}}\right)$ (4)
LP to HP	$H(s)$	$H\left(\dfrac{\omega_c}{s}\right)$	$H\left(t \dfrac{1 + z^{-1}}{1 - z^{-1}}\right)$ (5)
LP to BP	$H(s)$	$H\left(\dfrac{s^2 + \omega_L \omega_U}{(\omega_U - \omega_L)s}\right)$	$H\left(U \dfrac{1 - z^{-1}}{1 + z^{-1}} + L \dfrac{1 + z^{-1}}{1 - z^{-1}}\right)$ (6)
LP to BS	$H(s)$	$H\left(\dfrac{(\omega_U - \omega_L)s}{s^2 + \omega_L \omega_U}\right)$	$H\left(\dfrac{1}{U \dfrac{1 - z^{-1}}{1 + z^{-1}} + L \dfrac{1 + z^{-1}}{1 - z^{-1}}}\right)$ (7)

Where : $c = \cot\left(\pi \dfrac{f_c}{f_s}\right)$ $t = \tan\left(\pi \dfrac{f_c}{f_s}\right)$

$c_U = \cot\left(\pi \dfrac{f_U}{f_s}\right)$ $t_L = \tan\left(\pi \dfrac{f_L}{f_s}\right)$

$U = \dfrac{c_U}{1 - c_U t_L}$ $L = \dfrac{t_L}{1 - c_U t_L}$

f_s is the sampling frequency
f_c is the cut off frequency (Low pass and High pass)
f_U and f_L are the upper and lower frequency (Band pass and Band stop)

3 BILINEAR z-TRANSFORMATION WITH PASCAL'S TRIANGLE

This section introduces the bilinear z- transform with pre-warping to convert an analog transfer function H(s) to a digital transfer function H(z) involved the Pascal's triangle. With the Pascal's triangle, the matrix equation of the relationship between the digital coefficients and the analog coefficients can be found easy to compute and hand-calculated.

3.1 The Pascal's triangle

One of the most useful applications of the Pascal's triangle is to find coefficients and expand the binomial expression $(U \pm L)^n$ and it can be shown in fig.1 below.
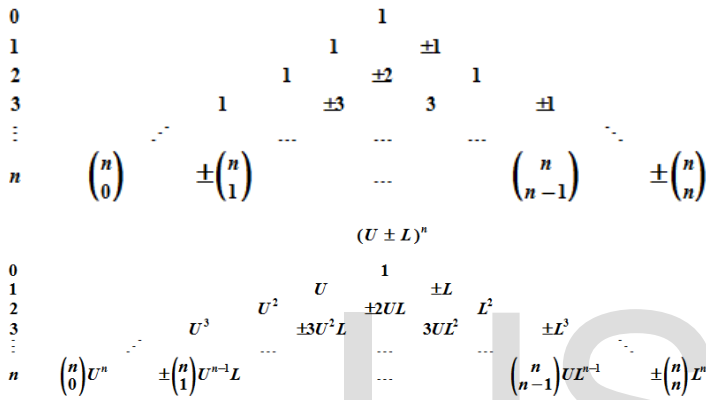

fig.1 the Pascal's triangle

$$(U+L)^n \Rightarrow \begin{cases} \text{The coefficients:} \binom{n}{i}_{i=0 \to n} & (8) \\ \text{The expansion:} \sum_{i=0}^{n}\binom{n}{i}U^{n-i}L^i & (9) \end{cases}$$

$$(U-L)^n \Rightarrow \begin{cases} \text{The coefficients:} (-1)^i\binom{n}{i}_{i=0 \to n} & (10) \\ \text{The expansion:} \sum_{i=0}^{n}(-1)^i\binom{n}{i}U^{n-i}L^i & (11) \end{cases}$$

The main key is addressed in the next sections considering the matrix equations of conversion from a low pass to low pass, high pass, band pass and band stop involved the Pascal's triangle.

3.2 Convert an analog low pass filter to a digital low pass filter

The transfer function of an nth-order low pass filter in s-domain H(s) and the transfer function of a low pass filter in z-domain H(z) can be described as follow:

$$H(s) = \frac{A_0 + A_1 s + ... + A_n s^n}{B_0 + B_1 s + ... + B_n s^n} = \frac{\sum_{i=0}^{n} A_i s^n}{\sum_{i=0}^{n} A_i s^n} \qquad (8)$$

$$H(z) = \frac{a_0 + a_1 z^{-1} + ... + a_n z^{-n}}{b_0 + b_1 z^{-1} + ... + b_n z^{-n}} = \frac{\sum_{i=0}^{n} a_i z^{-n}}{\sum_{i=0}^{n} b_i z^{-n}} \qquad (9)$$

Where A_i, B_i and a_i, b_i are all real coefficients in s-domain and z-domain respectively and n is a highest order number in the analog filter.
From equation (4), the matrix equations of the relationship

between a_i, A_i and b_i, B_i can be found as below

$$\begin{bmatrix} \left[(a_i)_{i=0 \to n}\right]_{n+1;1} = [P_{LP}]_{n+1;n+1}\left[(A_i c^i)_{i=1 \to n}\right]_{n+1;1} \\ \left[(a_i)_{i=0 \to n}\right]_{n+1;1} = [P_{LP}]_{n+1;n+1}\left[(A_i c^i)_{i=1 \to n}\right]_{n+1;1} \end{bmatrix} \quad (10)$$

All the numbers in the matrix $[P_{LP}]$ can be calculated from the positive and negative Pascal's triangle as five steps below.
Step1: All the numbers in the first row of $[P_{LP}]$ equal to 1. (This is the left edge of the Pascal's triangle)
Step 2: The numbers in the last row of $[P_{LP}]$ equal to $(-1)^i$. (This is the right edge of the Pascal's triangle)
Step 3: The numbers in the first column of $[P_{LP}]$ is the numbers in the nth row in the Pascal's triangle, equation (8).
Step 4: The numbers in the last column of $[P_{LP}]$ is the numbers in the nth row in the Pascal's triangle, equation (10).
Step 5: Another numbers in $[P_{LP}]$ can be computed following expression:

$$(P_{LP})_{i;j} = (P_{LP})_{i;j-1} - (P_{LP})_{i-1;j-1} - (P_{LP})_{i-1;j}$$

The matrix $[P_{LP}]$ of low pass to low pass can be rewritten as:

$$\left[(P_{LP})\right]_{n+1;n+1} = \begin{bmatrix} \begin{cases} (P_{LP})_{i=1; j=1 \to n+1} = 1 \\ (P_{LP})_{i=n+1; j=1 \to n+1} = (-1)^{j-1} \\ (P_{LP})_{i=1 \to n+1; j=1} = \binom{n}{i-1} \\ (P_{LP})_{i=1 \to n+1; j=n+1} = (-1)^{i-1}\binom{n}{i-1} \\ (P_{LP})_{i;j} = (P_{LP})_{i;j-1} - (P_{LP})_{i-1;j-1} - (P_{LP})_{i-1;j} \end{cases} \end{bmatrix}$$

Example 1: Convert a fourth-order Butterworth analog low pass filter has the transfer function H(s), the cut-off frequency of 200hz to digital low pass filter with fs=1kHz.

$$H(s) = \frac{1}{1 + 2.6131s + 3.4142s^2 + 2.6131s^3 + s^4}$$

$$n = 4 , \quad c = \cot(\pi\frac{f_c}{f_s}) = \cot(\pi\frac{200}{1000}) = 1.3764$$

$$A_0 = 1, A_1 = 0, A_2 = 0, A_3 = 0, A_4 = 0$$
$$B_0 = 1, B_1 = 02.6131, B_2 = 3.4142, B_3 = 2.6131, B_4 = 1$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 2 & 0 & -2 & -4 \\ 6 & 0 & -2 & 0 & 6 \\ 4 & -2 & 0 & 2 & -4 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 6 \\ 4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 2 & 0 & -2 & -4 \\ 6 & 0 & -2 & 0 & 6 \\ 4 & -2 & 0 & 2 & -4 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3.5967 \\ 6.4680 \\ 6.8136 \\ 3.5889 \end{bmatrix} = \begin{bmatrix} 21.4671 \\ -16.7893 \\ 14.5972 \\ -3.9215 \\ 0.6466 \end{bmatrix}$$

$$\therefore H(z) = \frac{1 + 4z^{-1} + 6z^{-2} + 4z^{-3} + z^{-4}}{21.4671 - 18.7893z^{-1} + 14.5972z^{-2} - 3.9215z^{-3} + 0.6466z^{-4}}$$
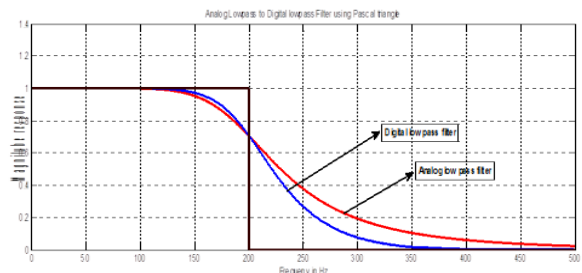
3.3 Convert an analog low pass filter to a digital high pass filter

The transfer function H(z) of the digital high pass filter can be written as the equation (9) and applying the equation (5), the matrix equation can described as equation (16) blow

$$\begin{cases} \left[(a_i)_{i=0\to n}\right]_{n+1;1} = \left[P_{LP}\right]_{n+1;n+1}\left[(A_i t^i)_{i=1\to n}\right]_{n+1;1} \\ \left[(a_i)_{i=0\to n}\right]_{n+1;1} = \left[P_{LP}\right]_{n+1;n+1}\left[(A_i t^i)_{i=1\to n}\right]_{n+1;1} \end{cases} \quad (11)$$

The matrix $[P_{HP}]$ can be found from the matrix $[P_{LP}]$ by swapping the first column to the last column, the second column to second last column and so on or can use the formula below

$$\left[(P_{LP})\right]_{n+1;n+1} = \begin{cases} (P_{LP})_{i=1;j=1\to n+1} = 1 \\ (P_{LP})_{i=n+1;j=1\to n+1} = (-1)^{j-1} \\ (P_{LP})_{i=1\to n+1;j=1} = (-1)^{i-1}\binom{n}{i-1} \\ (P_{LP})_{i=1\to n+1;j=n+1} = \binom{n}{i-1} \\ (P_{LP})_{i;j} = (P_{LP})_{i;j-1} + (P_{LP})_{i-1;j-1} + (P_{LP})_{i-1;j} \end{cases}$$

Example 2: Convert a fourth-order Butterworth analog low pass filter has the transfer function H(s), the cut-off frequency of 200hz to digital high pass filter with fs=1kHz.

$$H(s) = \frac{1}{1 + 2.6131s + 3.4142s^2 + 2.6131s^3 + s^4}$$

$$n = 4, \quad c = \tan(\pi\frac{f_c}{f_s}) = \tan(\pi\frac{200}{1000}) = 0.7265$$

$$A_0 = 1, A_1 = 0, A_2 = 0, A_3 = 0, A_4 = 0$$
$$B_0 = 1, B_1 = 02.6131, B_2 = 3.4142, B_3 = 2.6131, B_4 = 1$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & -2 & 0 & 2 & 4 \\ 6 & 0 & -2 & 0 & 6 \\ 4 & 2 & 0 & -2 & 4 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 6 \\ 4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & -2 & 0 & 2 & 4 \\ 6 & 0 & -2 & 0 & 6 \\ 4 & 2 & 0 & -2 & 4 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1.8985 \\ 1.8022 \\ 1.0022 \\ 0.2786 \end{bmatrix} = \begin{bmatrix} 5.9816 \\ -4.6782 \\ 4.0674 \\ -1.0927 \\ 0.1802 \end{bmatrix}$$

$$\therefore H(z) = \frac{1 + 4z^{-1} + 6z^{-2} + 4z^{-3} + z^{-4}}{5.9816 - 4.6782z^{-1} + 4.0674z^{-2} - 1.0927z^{-3} + 0.1802z^{-4}}$$
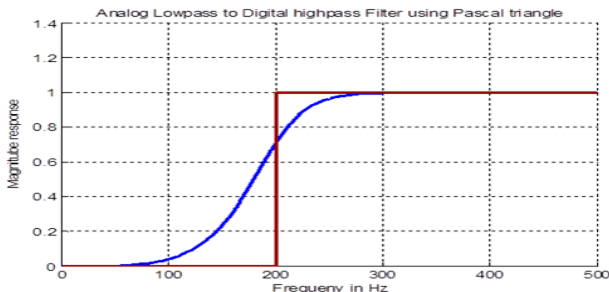
Converting a low pass to low pass and low pass to high pass were studied. The first and last column in the matrix $[P_{LP}]$ and $[P_{LP}]$ are the nth row in the Pascal's triangle. In the next sections, it is more interesting with Pascal's triangle to find the digital coefficients of converting low pass to band pass and band stop.

3.4 Convert an analog low pass filter to a digital band pass filter

The order of transfer function H(z) of the band pass filter is 2n, equation (12), it causes from the multiply of number 2 in s² with the nth-order when applying the frequency transformation from an nth-order low pass to a band pass filter. The transfer function H(z) and the matrix equation can be written as shown below:

$$H(z) = \frac{\sum_{i=0}^{2n} a_i z^{-i}}{\sum_{i=0}^{2n} b_i z^{-i}} \quad (12)$$

$$\begin{cases} \left[(a_i)_{i=0\to 2n;1}\right]_{2n+1;1} = \left[(P_{BP})_{i=1\to 2n+1;j=1\to 2n+1}\right]_{2n+1;2n+1}\left[\Delta_{BP}(A,U,L)_{i=0\to 2n;1}\right]_{2n+1;1} \\ \left[(b_i)_{i=0\to 2n;1}\right]_{2n+1;1} = \left[(P_{BP})_{i=1\to 2n+1;j=1\to 2n+1}\right]_{2n+1;2n+1}\left[\Delta_{BP}(B,U,L)_{i=0\to 2n;1}\right]_{2n+1;1} \end{cases} \quad (13)$$

The matrix $[P_{BP}]$ is the same with the matrix $[P_{HP}]$ but it has the size of (2n+1; 2n+1). From equation (6) and (12), the matrix $[\Delta_{BP}]$ can be found as figure.2 below
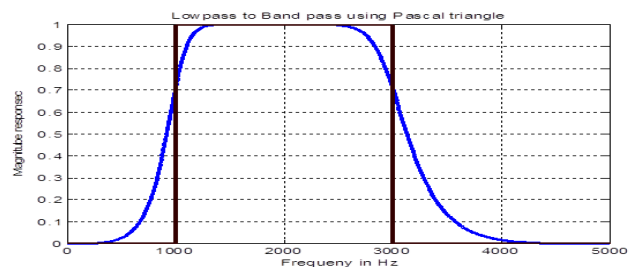

fig.2 the matrix $[\Delta]$ with the Pascal's triangle.

From the figure.2 above, if taking out the analog coefficient Ai in each column, the matrix $[\Delta_{BP}]$ is the Pascal's triangle expansion of $(U+L)^n$. And from that, a formula can be derived for the matrix $[\Delta BP]$ as shown

$$\begin{cases} \left[(\Delta_{BP}(A,U,L))_{i=0\to 2n;1}\right]_{2n+1;1} = \begin{cases} \left\{\sum_{k=1}^{S}\binom{R}{k-1}A_R U^{R-k+1}L^{k-1}\right\}_{i=0\to M} \\ \left\{\sum_{k=1}^{S'}\binom{R'}{k-1}A_R U^{R-k+1}L^{k-1}\right\}_{i=M+1\to 2n} \end{cases}_{2n+1;1} \\ \left[(\Delta_{BP}(B,U,L))_{i=0\to 2n;1}\right]_{2n+1;1} = \begin{cases} \left\{\sum_{k=1}^{S}\binom{R}{k-1}B_R U^{R-k+1}L^{k-1}\right\}_{i=0\to M} \\ \left\{\sum_{k=1}^{S'}\binom{R'}{k-1}B_R U^{R-k+1}L^{k-1}\right\}_{i=M+1\to 2n} \end{cases}_{2n+1;1} \end{cases}$$

Example 3: Convert a fourth-order Butterworth analog low pass filter has the transfer function H(s) to digital band pass filter with the lower frequency f_L =1 kHz, the upper frequency f_U=3 kHz and the sampling frequency fs =10 kHz.

$$H(s) = \frac{1}{1 + 2.6131s + 3.4142s^2 + 2.6131s^3 + s^4}$$

$$
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} =
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-8 & -6 & -4 & -2 & 0 & 2 & 4 & 6 & 8 \\
28 & 14 & 4 & -2 & -4 & -2 & 4 & 14 & 28 \\
-56 & -14 & 4 & 6 & 0 & -6 & -4 & 14 & 56 \\
70 & 0 & -10 & 0 & 6 & 0 & -10 & 0 & 70 \\
-56 & 14 & 4 & -6 & 0 & 6 & -4 & -14 & 56 \\
28 & -14 & 4 & 2 & -4 & 2 & 4 & -14 & 28 \\
-8 & 6 & -4 & 2 & 0 & -2 & 4 & -6 & 8 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1
\end{bmatrix}
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} =
\begin{bmatrix} 1 \\ 0 \\ -4 \\ 0 \\ 6 \\ 0 \\ -4 \\ 0 \\ 1 \end{bmatrix}
$$

$$
\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \\ b_8 \end{bmatrix} =
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-8 & -6 & -4 & -2 & 0 & 2 & 4 & 6 & 8 \\
28 & 14 & 4 & -2 & -4 & -2 & 4 & 14 & 28 \\
-56 & -14 & 4 & 6 & 0 & -6 & -4 & 14 & 56 \\
70 & 0 & -10 & 0 & 6 & 0 & -10 & 0 & 70 \\
-56 & 14 & 4 & -6 & 0 & 6 & -4 & -14 & 56 \\
28 & -14 & 4 & 2 & -4 & 2 & 4 & -14 & 28 \\
-8 & 6 & -4 & 2 & 0 & -2 & 4 & -6 & 8 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1
\end{bmatrix}
\begin{bmatrix} 0.8181 \\ 2.2479 \\ 4.5517 \\ 5.5011 \\ 4.7439 \\ 2.4602 \\ 0.9103 \\ 0.2011 \\ 0.0327 \end{bmatrix} =
\begin{bmatrix} 21.4671 \\ -39.2118 \\ 45.0596 \\ -39.8278 \\ 33.4033 \\ -19.0072 \\ 8.3336 \\ -2.4857 \\ 0.6466 \end{bmatrix}
$$

$$\therefore H(z) = \frac{1 - 4z^{-2} + 6z^{-4} - 4z^{-6} + z^{-8}}{21.4671 - 39.2118z^{-1} + 45.0596z^{-2} - 39.8278z^{-3} + 33.4033z^{-4} - 19.0072z^{-5} + 8.3336z^{-6} - 2.4857z^{-7} + 0.6466z^{-8}}$$

3.5 Convert an analog low pass filter to a digital band stop filter

The matrix equation for converting an analog low pass to digital band stop can be derived from equation (7).

$$
\begin{cases}
\left[(a_i)_{i=0 \to 2n\,;1}\right]_{2n+1\,;1} = \left[(P_{BS})_{i=1 \to 2n+1\,;\, j=1 \to 2n+1}\right]_{2n+1\,;\,2n+1} = \left[(\Delta_{BS}(A,U,L))_{i=0 \to 2n\,;1}\right]_{2n+1\,;1} \\
\left[(b_i)_{i=0 \to 2n\,;1}\right]_{2n+1\,;1} = \left[(P_{BS})_{i=1 \to 2n+1\,;\, j=1 \to 2n+1}\right]_{2n+1\,;\,2n+1} = \left[(\Delta_{BS}(B,U,L))_{i=0 \to 2n\,;1}\right]_{2n+1\,;1}
\end{cases} \quad (14)
$$

The matrix [P$_{BS}$] is the same the matrix [P$_{BP}$]. The matrix [Δ_{BS}] is similar with the matrix [Δ_{BP}], the only different is the analog coefficients A$_i$, B$_i$ is replaced by A$_{n-i}$ and B$_{n-i}$ and it can be shown as below.

$$
\begin{cases}
\left[(\Delta_{BP}(A,U,L))_{i=0 \to 2n\,;1}\right]_{2n+1\,;1} = \left\{ \begin{array}{l} \left(\sum_{k=1}^{S}\binom{R}{k-1}A_{R-M}U^{R-k+1}L^{k-1}\right)_{i=0 \to M} \\ \left(\sum_{k=1}^{S'}\binom{R}{k-1}A_{R-M}U^{R-k+1}L^{k-1}\right)_{i=M+1 \to 2n} \end{array} \right\}_{2n+1\,;1} \\
\left[(\Delta_{BP}(B,U,L))_{i=0 \to 2n\,;1}\right]_{2n+1\,;1} = \left\{ \begin{array}{l} \left(\sum_{k=1}^{S}\binom{R}{k-1}B_{R-M}U^{R-k+1}L^{k-1}\right)_{i=0 \to M} \\ \left(\sum_{k=1}^{S'}\binom{R}{k-1}B_{R-M}U^{R-k+1}L^{k-1}\right)_{i=M+1 \to 2n} \end{array} \right\}_{2n+1\,;1}
\end{cases}
$$

Example 4: Convert a fourth-order Butterworth analog low pass filter has the transfer function H(s) to digital band stop filter with the lower frequency f$_L$ =1 kHz, the upper frequency f$_U$=3 kHz and the sampling frequency f$_s$ =10 kHz.

$$H(s) = \frac{1}{1 + 2.6131s + 3.4142s^2 + 2.6131s^3 + s^4}$$

$$
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} =
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-8 & -6 & -4 & -2 & 0 & 2 & 4 & 6 & 8 \\
28 & 14 & 4 & -2 & -4 & -2 & 4 & 14 & 28 \\
-56 & -14 & 4 & 6 & 0 & -6 & -4 & 14 & 56 \\
70 & 0 & -10 & 0 & 6 & 0 & -10 & 0 & 70 \\
-56 & 14 & 4 & -6 & 0 & 6 & -4 & -14 & 56 \\
28 & -14 & 4 & 2 & -4 & 2 & 4 & -14 & 28 \\
-8 & 6 & -4 & 2 & 0 & -2 & 4 & -6 & 8 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1
\end{bmatrix}
\begin{bmatrix} 0.8181 \\ 0 \\ 1.4635 \\ 0 \\ 0.9818 \\ 0 \\ 0.2927 \\ 0 \\ 0.0327 \end{bmatrix} =
\begin{bmatrix} 3.5889 \\ -10.9666 \\ 26.9220 \\ -39.2997 \\ 47.8885 \\ -39.2997 \\ 26.9220 \\ -10.9666 \\ 3.5889 \end{bmatrix}
$$

$$
\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \\ b_8 \end{bmatrix} =
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-8 & -6 & -4 & -2 & 0 & 2 & 4 & 6 & 8 \\
28 & 14 & 4 & -2 & -4 & -2 & 4 & 14 & 28 \\
-56 & -14 & 4 & 6 & 0 & -6 & -4 & 14 & 56 \\
70 & 0 & -10 & 0 & 6 & 0 & -10 & 0 & 70 \\
-56 & 14 & 4 & -6 & 0 & 6 & -4 & -14 & 56 \\
28 & -14 & 4 & 2 & -4 & 2 & 4 & -14 & 28 \\
-8 & 6 & -4 & 2 & 0 & -2 & 4 & -6 & 8 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1
\end{bmatrix}
\begin{bmatrix} 0.8181 \\ 2.2479 \\ 4.5517 \\ 5.5011 \\ 4.7439 \\ 2.4602 \\ 0.9103 \\ 0.2011 \\ 0.0327 \end{bmatrix} =
\begin{bmatrix} 21.4671 \\ -39.2118 \\ 45.0596 \\ -39.8278 \\ 33.4033 \\ -19.0072 \\ 8.3336 \\ -2.4857 \\ 0.6466 \end{bmatrix}
$$

$$\therefore H(z) = \frac{3.5889 - 10.9666z^{-1} + 26.9220z^{-2} - 39.2997z^{-3} + 47.8885z^{-4} - 39.2997z^{-5} + 26.9220z^{-6} - 10.9666z^{-7} + 3.5889z^{-8}}{21.4671 - 39.2118z^{-1} + 45.0596z^{-2} - 39.8278z^{-3} + 33.4033z^{-4} - 19.0072z^{-5} + 8.3336z^{-6} - 2.4857z^{-7} + 0.6466z^{-8}}$$
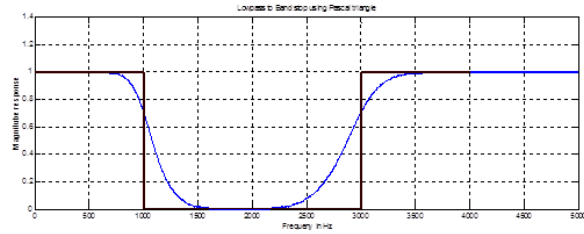
4 GENERAL FORMULA FOR CONVERTING AN ANALOG LOW PASS FILTER TO A DIGITAL FILTER

In the figuge.2, it can see that the Pascal's triangle involved into the matrix [Δ_{BP}] and is clearly by redraw it in table.2. The Δ_i is the sum of all the elements in the column of Pascal's triangle multiply with the analog coefficient at the same row.

n	Bp	Bs	(U + L)^n								
0	A$_0$	A$_n$				1					
1	A$_1$	⋮			U		L				
2	A$_2$	A$_3$		U^2		2UL		L^2			
3	A$_3$	A$_2$	U^3		3U^{2}L		3UL^2		L^3		
⋮	⋮	A$_1$	⋰	⋯	⋯	⋯	⋯	⋯	⋱		
n	A$_n$	A$_0$	U^n	⋯	⋯	⋯	⋯	⋯	⋯		L^n
LptoBp , LptoBs N=2n			Δ_0	Δ_1	⋯	⋯	Δ_{M-1}	Δ_M	Δ_{M+1}	⋯ ⋯	Δ_{2n-1} Δ_{2n}
			Δ_0	Δ_1	⋯	⋯	Δ_{n-1}	$\Delta_{M=n}$	LptoLp N=n , L = 0		
LptoHp N=n U = 0							Δ_0	Δ_1	⋯	⋯	Δ_{n-1} $\Delta_{M=n}$

Table.2 General formula of the matrix [Δ]

From the table.2 above, if let L=0, all the elements in Pascal's triangle have L will equal to zero, and the [Δ] becomes the matrix [Δ_{LP}] that is the left side edge of the Pascal's triangle. And if U=0, the [Δ] = [Δ_{HP}] is the right side edge of the Pascal's triangle. For the [Δ_{BS}], the analog coefficients A$_i$ and B$_i$ changes to A$_{n-i}$ and B$_{n-i}$. So the matrix [Δ_{BP}] can be used for all converting from Lp to Lp, Lp to Hp, Lp to Bp and Lp to Bs. And the general formula for all conversion is:

$$
\begin{cases}
\left[(a_i)_{i=0 \to N\,;1}\right]_{N+1\,;1} = [P]_{N+1\,;\,N+1}\left[(\Delta_{BS}(A,U,L))_{i=0 \to N+1\,;1}\right]_{N+1\,;1} \\
\left[(b_i)_{i=0 \to N\,;1}\right]_{N+1\,;1} = [P]_{N+1\,;\,N+1}\left[(\Delta_{BS}(B,U,L))_{i=0 \to N+1\,;1}\right]_{N+1\,;1}
\end{cases} \quad (15)
$$

5 THE PASCAL'S TRIANGLE WITH INSERTING ZEROS

$$
[T]_{n+1\,;\,N+1} =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & U & 0 & L & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & U^2 & 0 & 2UL & 0 & L^2 & 0 & 0 & 0 \\
0 & 0 & U^3 & 0 & 3U^2L & 0 & 3LU^2 & 0 & L^3 & 0 & 0 \\
\vdots & \ddots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \ddots & 0 \\
U^n & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & L^n
\end{bmatrix}
$$

figure.3 The Pascal's triangle with zeros

Inserting zeros into the Pascal's triangle creates a matrix [T] as in figure.3. From table.2 the Δi is the sum of all the elements in the column of Pascal's triangle multiply with the analog coefficient at the same row, it means that the matrix [Δ] can be found by matrix multiplication between matrix analog coefficients with the matrix [T] as shown below

$$\begin{cases} \left[\Delta(A,U,L) \right]_{1;N+1} = \left[A_i \right]_{1;n+1} \left[T \right]_{n+1;N+1} \\ \left[\Delta(B,U,L) \right]_{1;N+1} = \left[B_i \right]_{1;n+1} \left[T \right]_{n+1;N+1} \end{cases} \quad \text{(16)}$$

The equation (16) is another way to find the coefficients of a digital filter from the coefficients of an analog low pass filter and it can be used to invert from the coefficients of a digital filter to the coefficients of a low pass filter. The inverting of a digital filter to an analog low pass filter will be presented in the next paper "Transform a digital filter to another digital filter using Pascal's triangle".

6 CONCLUSION

In this work, the bilinear z-transform is used to convert an transfer function H(s) of an analog low pass to the transfer function H(z) of a digital filter and vice versa. The involving of the Pascal's triangle made both the direct and inverse transformation easier for computing and hand-calculated. And all the formulas are derived and demonstrated. It would be the powerful tools in using all these formulas for converting analog to digital and from digital to analog filter.

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